**1D Gaussian Integrals**

I’m mainly interested in path integrals vis a vis partition functions and correlations/green’s functions. But we’ll skip all that here and just give the facts…

**1 dimensional real Gaussian integrals**

Let’s consider the moment generating function of the Gaussian distribution exp[-(1/2)ax2], whose normalization we’ll just leave off. We have:



This can be worked out, and the result is:



So we have:



[Z(0) means setting j = 0] Let’s observe that the exponent at least can be ascertained as exponent of the integrand, evaluated at the minimum, which in this case is:



and the exponent evaluated there is:



So we can write:



This principle remains valid as we continue on to more complicated expressions. A Taylor series expansion of Z[j] will give us moments. We find, for even n [odd n gives us zero]:



The cumulant generating function is generally defined as W(j) = lnZ(j), and so we have:



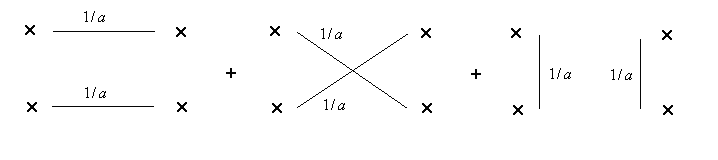
where for example <x>1 = <x>, <x>2 = <(x - <x>)2>, and I think <x>3 = <(x - <x>)3>, but after that the formulas get more complicated. In our case, we have:



So we see that all higher cumulants of Gaussian distribution just go to zero. Now let’s consider the graphical representation of correlation functions/averages. Consider for example <x4>. This is:



where Z(0) = √(2π/a) is the ‘normalization’ of the Gaussian distribution – as you can verify above. This can be represented graphically in the following way,



where each line connecting a pair of these points represents a propagator, G = <x2> = 1/a. Note there are 3 diagrams because given one x, there are three options for connecting it two another x. And once those two are connected, there are no more options left, except one, for connecting the remaining two. We can extend this to correlations of the form,



[Z(0) means setting j and λ = 0] We can say,



So we have:



There are three ways to get Dmn. First, we can just use the generating function to evaluate all of these integrals.



But while that’s easy to do for 1D Gaussian integrals. It’s not so easily generalizable to higher dimension integrals. The ‘better’ way is to construct the Feynman diagrams corresponding to p ‘external legs’, m ‘j-vertices’, and n ‘λ-vertices’. The Feynman diagram parts corresponding to these are:

A picture containing chart

Description automatically generated

And our diagram with p ‘external points’, m ‘j-vertices’, and n ‘λ-vertices’ will consist of p, m, and n of those aforementioned Feynman diagram parts, all joined together in the topologically distinct ways possible. Each λ vertex in the FD will carry a factor of λ, each j-vertex will carry a factor of j, and each connected line/propagator will carry a factor of 1/a. Then we can calculate Dmn in terms of these Feynman diagrams,



where the Multiplicity is the number of ways to construct such a diagram from the parts. And the Symmetry Factor is, well Multiplicity/(3!)nm!n!. As a check, we might note that comparing our explicit formula for Dmn and the Multiplicity formula for Dmn, we have:



The Feynman rules for the symmetry factors are:



Will do an example,

***example***

So consider,



We know this is:



And diagramatically, this would be, out to 6th order (odd orders give zero – there’s no way to construct such a diagram anyway)

A picture containing clock, antenna

Description automatically generated

O(2) Multiplicity is 1, as there is only one way to connect to j-vertices. Symmetry factor is (1/2!)! for the two indistinguishable vertices.

O(4) Multiplicity is 3, as we have 3 ways to connect the first j, and then we have only two j’s left, and so no more options. Symmetry factor is (1/2!)(1/2!)2 for 2 sets of indistinguishable vertices and the 2 pairs of indistinguishable vertices.

O(6) Multiplicity is 5·3, as we have five options for the first connection, and three for the second, but none for the third. Symmetry factor is (1/3!)(1/2!)3 for the 3 sets of indistinguishable vertices, and the 3 pairs of indistinguishable vertices.

So, we have, calculating both ways, just to see they’re the same:



So up to this order we have:



and we can see this is working out to exp(j2/2a).

***example***

Now consider following integral,



Out to second order in λ (first non-vanishing order) this would be:

Diagram

Description automatically generated

Now the first O(2) diagram has a multiplicity of 3·3 because there are three ways to choose the leg on the first λ and three ways to choose the leg on the second λ, which connect. Then by default the other two legs on each will connect. The symmetry factor would be (1/2)2 for two coincident propagators, and (1/2!) for two identical vertices. And second diagram has a multiplicity of 3·2 since there are three ways to connect a leg of the first λ to the second λ, and then two ways to connect a remaining leg of the first lambda to the second lambda. And then default connect the remaining. And it gets symmetry factor of (1/3!) for three identical propagators, and (1/2!) for two identical vertices. And this works out to, doing the two different ways:



So to second order in λ, we have:



And we’ll note that the total multiplicity of the O(2) diagrams is 3·3 + 3·2 = 15, which equals (p + m + 3n - 1)!! = (0 + 0 + 3·2 - 1)!! = 5!!.

**‘1 dimensional’ complex Gaussian integrals**

Now consider a complex Gaussian distribution. This is basically just a 2D Gaussian distribution in terms of (x,y), but which will be put in terms of (z,z\*) instead, by change of variables. The moment generating function is:



Note this is actually the square of the previous result. We can prove this as follows:



(Jacobian of transformation is 2 so,) So we have:



Again we’ll observe that evaluating the exponent at the minimum gives us the correct e() factor,



So we can write:



A Taylor expansion in powers of j, j\* would give the moments:



Well, we can work this expectation out. Note m = n necessarily for a non-zero result, as otherwise we have, in real space, an odd number of x’s or y’s.



The cumulant generating function is given by W(j,j\*) = lnZ(j,j\*). So we have in general:



and of course derivatives w/r to j, j\* give the cumulants…not sure how a multivariable cumulant is defined, maybe, e.g., <zz\*>1,2 = <z(z\* - <z\*>)2> or something. And in our case here,



Again we can represent various correlations/averages graphically. Consider first the results:



where Z(0,0) = 2π/a is the normalization of our distribution. And also consider:



then we could represent the latter diagramatically as

Chart

Description automatically generated with medium confidence

where the arrows go from \* to non -\*. Note that if we don’t have the same number of z’s and z\*’s, then the integral is zero (just like if we don’t have the same number of ψ’s, and ψ+’s). If we want to calculate something like (well λ would have to be negative for this to converge, or otherwise positive but at least small, and we’d have to keep the range of z restricted in this case so integrand doesn’t blow up),



[Z(0) means setting j and λ = 0] This can be calculated via the same perturbative scheme. We can say,



So we have:



Again, we could just evaluate the integral, and we’d get:



But also again, we typically won’t be able to explicitly evaluate all integrals like this, when we go to higher dimensions. So we’ll go the Feynman diagram route again. The diagram parts are now:

A group of airplanes flying in the sky

Description automatically generated with low confidence

And we can write Dmm´n as:



where the Multiplicity is the number of ways to construct such a diagram from the parts. And the Symmetry Factor is, well Multiplicity/(2!2!)nm!m´!n!. As a check, we might note that comparing our explicit formula for Dmn and the Multiplicity formula for Dmn, we have:



The Feynman rules for the symmetry factors are:



(there is no ½ for each coincident propagator because unlike for real numbers, there is only one direction/way to join a z and z\* together) Let’s do an example,

**example**

okay. Let’s do,



out to second order. The O(0) result is of course 1. Next,

Shape, circle

Description automatically generated

Apropos our O(1) diagram, the multiplicity would be 2, as we have two choices in connecting a given leg to another, and then the other two are connected by default. Our symmetry factor is (1/2!) as we have two indistinguishable propagators. So



That matches, as it should. And to next order, we have three topologically distinct diagrams:

Shape

Description automatically generated

The first has a multiplicity (4)(4), for the four choices of which two legs (out of z1z1\*z2z2\*) connect amongst themselves on the left λ (z1z1\*, z1z2\*, z2z1\*, z2z2\*) and same four choices of which two legs connect amongst themselves on the right λ; once these have been decided, then there no remaining choices for the ways to connect the vertices together. And it has a symmetry factor of (1/2!) two indistinguishable verticies. Note the propagators connecting the vertices are *distinguishable* b/c their arrows are going in opposite directions. The second diagram has multiplicity of (2)(2), and a symmetry factor of (1/2!)(1/2!)(1/2!) for two indistinguishable vertices, and two indistinguishable propagators on the left part and another two indistinguishable propagators on the right part. And the third has a multiplicity of (2)(2), and a symmetry factor of (1/2!)(1/2!)(1/2!) for two indistinguishable vertices, and two indistinguishable propagators (ones going left→right), and another two indistinguishable propagators (ones going right → left). Observe the overall multiplicity of these diagrams is (p+m+2n)! = (0 + 0 +2·2)! = 24, which is what we just found (16 + 4 + 4). So these diagrams give us:



That’s reassuring. So out to second order, we have:



If we do a direct Taylor series expansion, we’d have:



So that matches!

**1D Grassman Guassian integrals**

Grassman integrals are somewhat easy to integrate due to the fact that integrands are at most linear. I think we can take to be basically ψ\*, and same with ~ j\*. But note j is a Grassman number, along with ψ.



(A is real I think, and so A *commutes* with the ψ’s) So what is this? Well first let’s observe,



And now,



(note sure we can factor the A out like that, but…) So we have:



Again we’ll observe that evaluating the exponent at the minimum gives us the correct e() factor. Remember ψ commutes with A b/c A is complex (or real whatever; it’s not Grassman).



So we can write:



A Taylor expansion in powers of j, would give the moments:



As before, m = n necessarily for a non-zero result. And unlike before only the m = n = 0 or 1 terms will be non-zero.



Since we can’t have correlations beyond this, at the moment, I guess there’s no point in developing a diagrammatic apparatus. We’ll do this in next file.